Module 2 – Electricity Spot Markets (e.g. day-ahead)

2.2 Market clearing as an optimization problem

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Introducing notations first

**Inputs:**

- All offers in the market are formulated in terms of a *quantity* \( P \) and a *price* \( \lambda \)

- On the *supply* side (\( N_G \) supply offers):
  - set of offers: \( \mathcal{L}_G = \{ G_j, \ j = 1, \ldots, N_G \} \)
  - maximum quantity for offer \( G_j \): \( P_G^j \)
  - price for offer \( G_j \): \( \lambda_G^j \)

- On the *demand* side (\( N_D \) demand offers):
  - set of offers: \( \mathcal{L}_D = \{ D_i, \ i = 1, \ldots, N_D \} \)
  - maximum quantity for offer \( D_i \): \( P_D^i \)
  - price for offer \( D_i \): \( \lambda_D^i \)

**Decision variables:**

- *Generation* schedule: \( y^G = [y_1^G, \ldots, y_{N_G}^G]^\top, 0 \leq y_j^G \leq P_G^j \)
- *Consumption* schedule: \( y^D = [y_1^D, \ldots, y_{N_D}^D]^\top, 0 \leq y_i^D \leq P_D^i \)
Our example auction setup

*Supply*: (for a total of 1435 MWh)

<table>
<thead>
<tr>
<th>Company</th>
<th>Supply/Demand</th>
<th>id</th>
<th>$P_j^G$ (MWh)</th>
<th>$\lambda_j^G$ (€/MWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RT®</td>
<td>Supply</td>
<td>$G_1$</td>
<td>120</td>
<td>0</td>
</tr>
<tr>
<td>WeTrustInWind</td>
<td>Supply</td>
<td>$G_2$</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>BlueHydro</td>
<td>Supply</td>
<td>$G_3$</td>
<td>200</td>
<td>15</td>
</tr>
<tr>
<td>RT®</td>
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<td>$G_4$</td>
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<td>30</td>
</tr>
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<td>$G_5$</td>
<td>60</td>
<td>32.5</td>
</tr>
<tr>
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<td>50</td>
<td>34</td>
</tr>
<tr>
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<td>Supply</td>
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<td>60</td>
<td>36</td>
</tr>
<tr>
<td>DirtyPower</td>
<td>Supply</td>
<td>$G_8$</td>
<td>100</td>
<td>37.5</td>
</tr>
<tr>
<td>DirtyPower</td>
<td>Supply</td>
<td>$G_9$</td>
<td>70</td>
<td>39</td>
</tr>
<tr>
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<td>Supply</td>
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<td>40</td>
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<tr>
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<td>60</td>
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<td>$G_{12}$</td>
<td>45</td>
<td>70</td>
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<tr>
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<td>$G_{14}$</td>
<td>60</td>
<td>150</td>
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<tr>
<td>SafePeak</td>
<td>Supply</td>
<td>$G_{15}$</td>
<td>50</td>
<td>200</td>
</tr>
</tbody>
</table>
Our example auction setup

*Demand:* (for a total of 1065 MWh)

<table>
<thead>
<tr>
<th>Company</th>
<th>Supply/Demand</th>
<th>id</th>
<th>$P^D_i$ (MWh)</th>
<th>$\lambda^D_i$ (€/MWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CleanRetail</td>
<td>Demand</td>
<td>$D_1$</td>
<td>250</td>
<td>200</td>
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<tr>
<td>El4You</td>
<td>Demand</td>
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<td>EVcharge</td>
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<td>100</td>
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<td>QualiWatt</td>
<td>Demand</td>
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<td>90</td>
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<td>IntelliWatt</td>
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<td>El4You</td>
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<td>CleanRetail</td>
<td>Demand</td>
<td>$D_7$</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>IntelliWatt</td>
<td>Demand</td>
<td>$D_8$</td>
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<td>40</td>
</tr>
<tr>
<td>QualiWatt</td>
<td>Demand</td>
<td>$D_9$</td>
<td>30</td>
<td>38</td>
</tr>
<tr>
<td>IntelliWatt</td>
<td>Demand</td>
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<td>31</td>
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<tr>
<td>CleanRetail</td>
<td>Demand</td>
<td>$D_{11}$</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>El4You</td>
<td>Demand</td>
<td>$D_{12}$</td>
<td>10</td>
<td>16</td>
</tr>
</tbody>
</table>

That is a lot of offers to match... Could an optimization problem readily give us the solution?
Centralized social welfare optimization

- The *social welfare maximization* problem can be written as

\[
\begin{align*}
\text{max}_{y^G, y^D} & \quad \sum_{i=1}^{N_D} \lambda_i^D y_i^D - \sum_{j=1}^{N_G} \lambda_j^G y_j^G \\
\text{subject to} & \quad \sum_{j=1}^{N_G} y_j^G - \sum_{i=1}^{N_D} y_i^D = 0 \\
& \quad 0 \leq y_i^D \leq P_i^D, \ i = 1, \ldots, N_D \\
& \quad 0 \leq y_j^G \leq P_j^G, \ j = 1, \ldots, N_G
\end{align*}
\] (1a)

- And equivalently as a *minimization problem* by minimizing the opposite objective function, i.e.

\[
\begin{align*}
\text{min}_{y^G, y^D} & \quad \sum_{j=1}^{N_G} \lambda_j^G y_j^G - \sum_{i=1}^{N_D} \lambda_i^D y_i^D \\
\text{subject to} & \quad (1b)-(1d)
\end{align*}
\] (2a)
It is a simple linear program!

- One recognize a so-called **Linear Program** (LP, here in a compact form):

\[
\begin{align*}
\text{min}_{y} & \quad c^\top y \\
\text{subject to} & \quad Ay \leq b \\
& \quad A_{eq}y = b_{eq} \\
& \quad y \geq 0 
\end{align*}
\]

- LP problems can be readily solved in
  - **Matlab**, for instance with the function `linprog`,
  - **R**, with the library/function `lp.solve`,
  - and also obviously with **GAMS**, **Gurobi**, etc.

- However, for e.g. **R** and **Matlab**, you need to know how to build relevant vectors and matrices

- And, the solution will only give you the energy schedules in terms of supply and demand
Vector and matrices in the objective function

- The vector $y$ of optimization variables $c$ of weights in the objective function are constructed as

$$y = \begin{bmatrix} y_1^G \\ y_2 \\ \vdots \\ y_{N_G}^G \\ y_1^D \\ y_2^D \\ \vdots \\ y_{N_D}^D \end{bmatrix}, \quad y \in \mathbb{R}^{(N_G+N_D)}$$

$$c = \begin{bmatrix} \lambda_1^G \\ \lambda_2 \\ \vdots \\ \lambda_{N_G}^G \\ -\lambda_1^D \\ -\lambda_2^D \\ \vdots \\ -\lambda_{N_D}^D \end{bmatrix}, \quad c \in \mathbb{R}^{(N_G+N_D)}$$
Vector and matrices defining constraints

- For the equality constraint (balance of generation and consumption):
  \[ A_{eq} = [1 \ldots 1 - 1 \ldots -1], \quad A_{eq} \in \mathbb{R}^{(N_G+N_D)}, \quad b_{eq} = 0 \]

- For the inequality constraint (i.e., generation and consumption levels within limits):
  \[ A = \begin{bmatrix}
    1 & & & & \\
    & \ddots & & & \\
    & & 1 & & \\
    & & & 0 & \\
    0 & & & & 1
  \end{bmatrix}, \quad b = \begin{bmatrix}
    P^G_1 \\
    P^G_2 \\
    \vdots \\
    P^G_{N_G} \\
    P^D_1 \\
    P^D_2 \\
    \vdots \\
    P^D_{N_D}
  \end{bmatrix}, \quad \text{with } A \in \mathbb{R}^{(N_G+N_D) \times (N_G+N_D)} \text{ and } b \in \mathbb{R}^{(N_G+N_D)}

- Do not forget the non-negativity constraints for the elements of \( y \)...
Getting the complete market-clearing

- By complete market-clearing is meant obtaining
  - the schedule for all supply and demand offers, as well as
  - the price at which the market is cleared, i.e., the so-called market-clearing or system price (in, e.g., Nord Pool)

\[ \lambda, \nu \]

\[ \lambda = \lambda_S \nu = [\nu_G^1, ..., \nu_G^N]^\top \nu_D^1, ..., \nu_D^N]^\top \]

Getting the complete market-clearing

- By *complete* market-clearing is meant obtaining
  - the *schedule* for all supply and demand offers, as well as
  - the *price* at which the market is cleared, i.e., the so-called *market-clearing* or *system* price (in, e.g., Nord Pool)

- The system price is obtained through the dual of the LP previously defined, i.e.,

\[
\begin{align*}
\max_{\lambda, \nu} & \quad - b^T \nu \\
\text{subject to} & \quad A_{eq}^T \lambda - A^T \nu \leq c \\
& \quad \nu \geq 0
\end{align*}
\]

- This is also an LP: it can be solved with Matlab, R, GAMS, etc.

- \( \lambda \) and \( \nu \) are sets of *Lagrange multipliers* associated to all *equality* and *inequality* constraints:

\[
\lambda = \lambda^S \\
\nu = [\nu_1^G \ldots \nu_{N_g}^G \nu_1^D \ldots \nu_{N_d}^D]^T
\]

More specifically for the market-clearing problem

- Only one equality constraint, i.e.,

\[ \sum_{i} y_i^D - \sum_{j} y_j^G = 0 \]

for which the associated Lagrange multiplier \( \lambda^S \) represents the system price.
More specifically for the market-clearing problem

- Only one **equality** constraint, i.e.,
  \[ \sum_i y_i^D - \sum_j y_j^G = 0 \]
  for which the associated Lagrange multiplier \( \lambda^S \) represents the system price.

- And \( N_D + N_G \) **inequality** constraints:
  \[ 0 \leq y_i^D \leq P_i^D, \quad i = 1, \ldots, N_D, \quad 0 \leq y_j^G \leq P_j^G, \quad j = 1, \ldots, N_G \]
  for which the associated Lagrange multipliers \( \nu_i^D \) and \( \nu_j^G \) represents the unitary benefits for the various demand and supply offers if the market is cleared at \( \lambda^S \).
More specifically for the market-clearing problem

- Only one **equality** constraint, i.e.,
  \[ \sum_{i} y_{i}^{D} - \sum_{j} y_{j}^{G} = 0 \]
  for which the associated Lagrange multiplier \( \lambda^{S} \) represents the system price.

- And \( N_{D} + N_{G} \) **inequality** constraints:
  \[ 0 \leq y_{i}^{D} \leq P_{i}^{D}, \quad i = 1, \ldots, N_{D} \]
  \[ 0 \leq y_{j}^{G} \leq P_{j}^{G}, \quad j = 1, \ldots, N_{G} \]
  for which the associated Lagrange multipliers \( \nu_{i}^{D} \) and \( \nu_{j}^{G} \) represents the unitary benefits for the various demand and supply offers if the market is cleared at \( \lambda^{S} \).

- The dual of the market clearing LP is also an LP which writes
  \[
  \max_{\lambda^{S}, \{\nu_{i}^{D}\}, \{\nu_{j}^{G}\}} \quad \sum_{j} \nu_{j}^{G} P_{j}^{G} - \sum_{i} \nu_{i}^{D} P_{i}^{D}
  \]
  subject to
  \[
  \lambda^{S} - \nu_{j}^{G} \leq \lambda_{j}^{G}, \quad j = 1, \ldots, N_{G} \\
  - \lambda^{S} - \nu_{i}^{D} \leq -\lambda_{i}^{D}, \quad i = 1, \ldots, N_{D} \\
  \nu_{j}^{G} \geq 0, \quad j = 1, \ldots, N_{G}, \quad \nu_{i}^{D} \geq 0, \quad i = 1, \ldots, N_{D}
  \]

[To retrieve the dual LP, follow: Lahaie S (2008). How to take the dual of a Linear Program. (link)]
Let’s also write it as a compact linear program!

As for the **primal LP** allowing to obtain the dispatch for market participants on both supply and demand side, we write here the **dual LP** in a compact form:

\[
\begin{align*}
\text{max} & \quad \tilde{c}^\top \tilde{y} \\
\text{subject to} & \quad \tilde{A} \tilde{y} \leq \tilde{b} \\
& \quad \tilde{y} \geq 0
\end{align*}
\]

The next 2 slides describe how to build the assemble the relevant vectors and matrices in the above LP...

Then, it can be solved with **Matlab**, **R**, **GAMS**, etc.

And, the solution will give you the equilibrium price, as well as the unit benefits for each and every market participant.

[**NB**: Most optimization functions and tools readily give you the solution of dual problems when solving the primal ones! E.g., see documentation of \texttt{linprog} in Matlab]
Vector and matrices in the objective function

- The vector $\mathbf{y}$ of optimization variables $\mathbf{c}$ of weights in the objective function are constructed as

$$\tilde{\mathbf{y}} = \begin{bmatrix}
\nu_1^G \\
\nu_2^G \\
\vdots \\
\nu_{N_G}^G \\
\nu_1^D \\
\nu_2^D \\
\vdots \\
\nu_{N_D}^D \\
\lambda_S
\end{bmatrix}, \quad \tilde{\mathbf{y}} \in \mathbb{R}^{(N_G+N_D+1)}$$

$$\tilde{\mathbf{c}} = \begin{bmatrix}
-P_1^G \\
-P_2^G \\
\vdots \\
-P_{N_G}^G \\
-P_1^D \\
-P_2^D \\
\vdots \\
-P_{N_D}^D \\
0
\end{bmatrix}, \quad \tilde{\mathbf{c}} \in \mathbb{R}^{(N_G+N_D+1)}$$
Vector and matrices defining constraints

- No equality constraint!

- For the inequality constraint:

\[
\tilde{A} = \begin{bmatrix}
-1 & \cdots & -1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -1 \\
\end{bmatrix},
\tilde{b} = \begin{bmatrix}
\lambda^G_1 \\
\lambda^G_2 \\
\vdots \\
-\lambda^D_1 \\
-\lambda^D_2 \\
\vdots \\
-\lambda^D_{N_D}
\end{bmatrix},
\]

with \( \tilde{A} \in \mathbb{R}^{(N_G+N_D) \times (N_G+N_D)} \) and \( \tilde{b} \in \mathbb{R}^{N_G+N_D} \)
Application to our simple auction example

- Solving the **primal LP** for obtaining the supply and demand schedules yields:

<table>
<thead>
<tr>
<th>Supply id.</th>
<th>Schedule (MWh)</th>
<th>Demand id.</th>
<th>Schedule (MWh)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G₁</td>
<td>120</td>
<td>D₁</td>
<td>250</td>
</tr>
<tr>
<td>G₂</td>
<td>50</td>
<td>D₂</td>
<td>300</td>
</tr>
<tr>
<td>G₃</td>
<td>200</td>
<td>D₃</td>
<td>120</td>
</tr>
<tr>
<td>G₄</td>
<td>400</td>
<td>D₄</td>
<td>80</td>
</tr>
<tr>
<td>G₅</td>
<td>60</td>
<td>D₅</td>
<td>40</td>
</tr>
<tr>
<td>G₆</td>
<td>50</td>
<td>D₆</td>
<td>70</td>
</tr>
<tr>
<td>G₇</td>
<td>60</td>
<td>D₇</td>
<td>60</td>
</tr>
<tr>
<td>G₈</td>
<td>55</td>
<td>D₈</td>
<td>45</td>
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<tr>
<td>G₉-G₁₅</td>
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<td>D₉</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D₁₀-D₁₂</td>
<td>0</td>
</tr>
</tbody>
</table>

for a total amount of energy scheduled of 995 MWh

- Solving the **dual LP** gives a system price of 37.5 €/MWh which corresponds to the price offer of $G₈$
Use the self-assessment quizz to check your understanding!